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13. ABSTRACT *(Maximum 200 words)*

This work was concerned with research on fundamental mechanics and mathematics of *large deformation induced failures* in nonlinear solids. The specific area investigated was that of *void nucleation and growth* due to large deformations in nonlinear solids. Research on cavitation phenomena, which serve as a precursor to *fracture*, is crucial to the understanding of *failure mechanisms* in rubber-like solids (e.g. polymers, solid rocket propellants) and of ductile fracture processes in metals. Mathematically, the work involves investigation of singular solutions of the second-order quasilinear system of partial differential equations describing equilibrium states of nonlinearly elastic bodies. For radially symmetric deformations, the basic problem reduces to a bifurcation problem for a single second-order nonlinear ordinary differential equation. Particular emphasis was placed on the effect of *material inhomogeneity, compressibility and anisotropy* on void nucleation and growth.
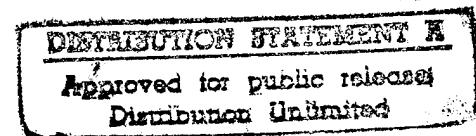
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FINAL REPORT

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Title: Large Deformation Failure Mechanisms in Nonlinear Solids

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1. Summary of Research Results

This work was concerned with research on the fundamental mechanics and mathematics of material failure mechanisms in nonlinear solids and structures. The specific area investigated was that of void nucleation and growth due to large deformations in nonlinear solids. Research on cavitation phenomena, which serve as a precursor to fracture, is crucial to the understanding of failure mechanisms in rubber-like solids (e.g. polymers, solid rocket propellants, aircraft tires) and of ductile fracture processes in metals. Mathematically, the work involved investigation of singular solutions of the partial differential equations describing equilibrium states of non-linearly elastic bodies. For radially symmetric deformations, the basic problem reduces to a bifurcation problem for a single second-order nonlinear ordinary differential equation. Particular emphasis was placed on the effect of material inhomogeneity, compressibility and anisotropy on void nucleation and growth. Studies on the micromechanics of particles, in particular the correlation of inclusions to void formation, are receiving much attention from the solid mechanics, applied mathematics and materials science communities. This research emphasis continues to be highlighted in National Committee reports on Solid Mechanics and Material Science Research Directions. The work also has impact on failure mechanisms due to large deformations in anisotropic and composite materials. Compared to the vast amount of information available on small deformations of such materials, results on large deformations remain virtually unexplored. Considerations of large deformations in anisotropic or composite materials often lead to striking differences from predictions of corresponding linearized theories. In view of the rapid utilization of advanced composite materials in present day Air Force technology, studies on the fundamental mechanics and mathematics of large deformations of such materials promise to have widespread impact on the AFOSR mission.

In what follows, we summarize briefly the main results obtained in the list of publications [1-13] attached. Detailed abstracts of each paper are given with the list of publications.

Cavity nucleation involving the formation and subsequent growth and coalescence of voids from microscopic inclusions leading to ductile fracture in metals and alloys has long been of concern to metallurgists. The phenomenon of sudden void formation ("cavitation") also arises in rubber elasticity and is also of interest in soil mechanics in connection with pile driving. Because of the role such cavitation phenomena play in failure processes for metals and polymers, such problems have attracted much recent attention from the solid mechanics and applied mathematics communities.

The impetus for much of the recent theoretical developments has been supplied by the mathematical work of Ball in 1982 concerned with *singular* solutions in nonlinear elasticity. Ball has studied a class of *bifurcation problems* for the equations of nonlinear elasticity which model the appearance of a cavity in the interior of an apparently solid homogeneous isotropic elastic body once a critical load has been attained. The alternative interpretation for such problems in terms of the growth of a *pre-existing* micro-void is more attractive from a physical point of view. It is important for us to emphasize here that the idealized mathematical treatment using a bifurcation approach does correctly predict the critical load at which a *pre-existing* microvoid will undergo sudden rapid growth. The critical load is automatically given by this approach - imposition of a failure criterion is not necessary. The bifurcation approach to cavitation may thus be viewed as a convenient analytical tool which furnishes values of the critical load. A treatment of the pre-existing traction-free microvoid problem is more complicated analytically. It is also important to note that *infinitesimal* theories of isotropic solid mechanics (including the classical theories of small strain plasticity) do *not* predict this cavitation phenomenon.

In [1], the effect of material *anisotropy* was examined. Very little is known about large deformations of anisotropic nonlinearly elastic materials. In [1] we were able to obtain closed-form analytic solutions to a cavitation problem for a transversely-isotropic incompressible non-linearly elastic sphere. One of the striking results found was that the bifurcation could be either to the right (supercritical) or to the left (subcritical) *depending on the degree of anisotropy* of the material. In the latter case, the cavity has finite radius on first appearance. This discontinuous change in stable equilibrium configurations ("snap cavitation") is reminiscent of the snap-through buckling phenomenon observed in certain structural mechanics problems. A similar snap cavitation phenomenon has been encountered by Antman and Negron-Marrero in the study of radially symmetric equilibrium states of *homogeneous anisotropic compressible* nonlinearly elastic bodies, and by Horgan and Pence for *composite* incompressible *isotropic* materials. Such a striking material instability does *not* occur for homogeneous isotropic materials. The radial anisotropy considered in [1] arises in material processing, for example, due to the thermal gradients arising in metal casting. See the attached figures from Walker, J. L., Structure of ingots and castings, 1958. Our results show that such anisotropy can lead to material failure.

The effects of material anisotropy and inhomogeneity on void nucleation and growth in incompressible anisotropic nonlinearly elastic solids are examined in [2]. A bifurcation problem is considered for a *composite* sphere composed of two arbitrary homogeneous incompressible nonlinearly elastic materials which are transversely isotropic about the radial direction, and per-fectly bonded across a spherical interface. Under a uniform radial tensile dead-load, a branch of radially symmetric configurations involving a traction-free internal cavity bifurcates from the
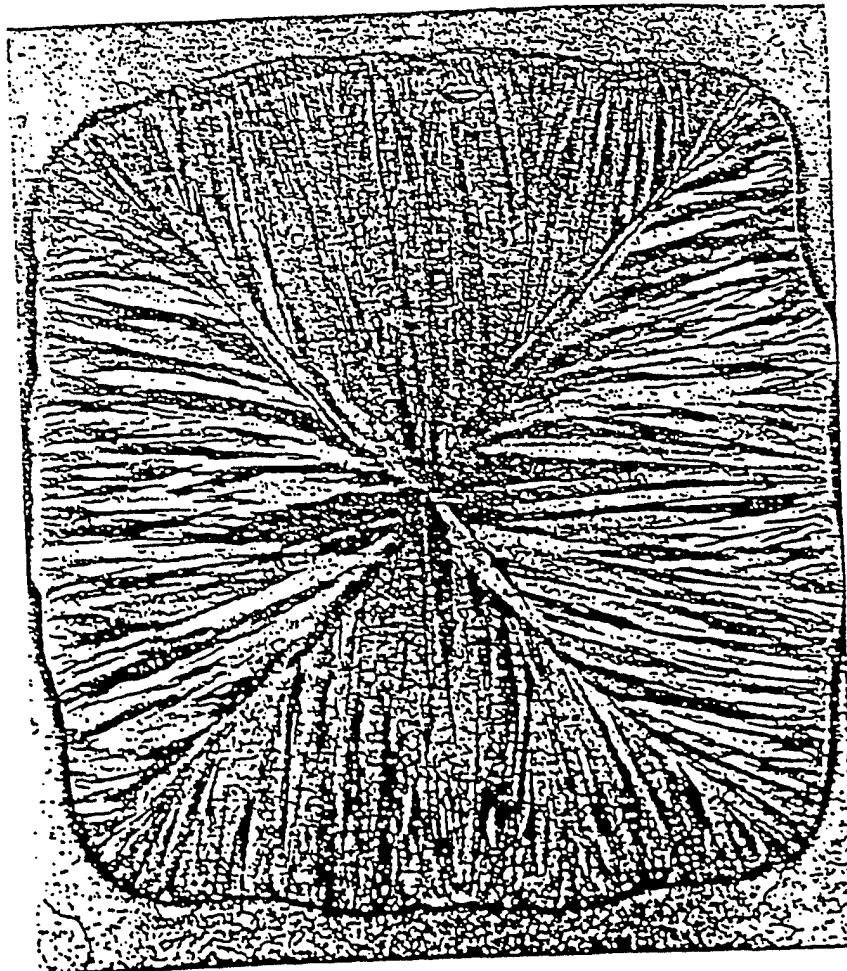
Fig. 2a—Cross Section from 5 inch X 5 inch Stainless Steel Ingot.

in the liquid at the time that the transformation is occurring. The growth rate in pure supercooled liquids can be described by (3)

$$G' = A\Delta T^{...}$$

in which A is roughly proportional to the fluidity and ΔT is the supercooling at the solid-liquid interface. Hillig and Turnbull (3) have explained the temperature dependence of G in terms of growth occurring by a screw dislocation mechanism. In metal systems undergoing rapid freezing, the latent heat of fusion heats the interface to the temperature at which the supercooling is sufficient to maintain growth at a rate consistent with the heat extraction rate. The rate of growth then depends on the rate at which the latent heat of fusion can be dissipated. Because of the heat transfer dependence of the rate of growth, the
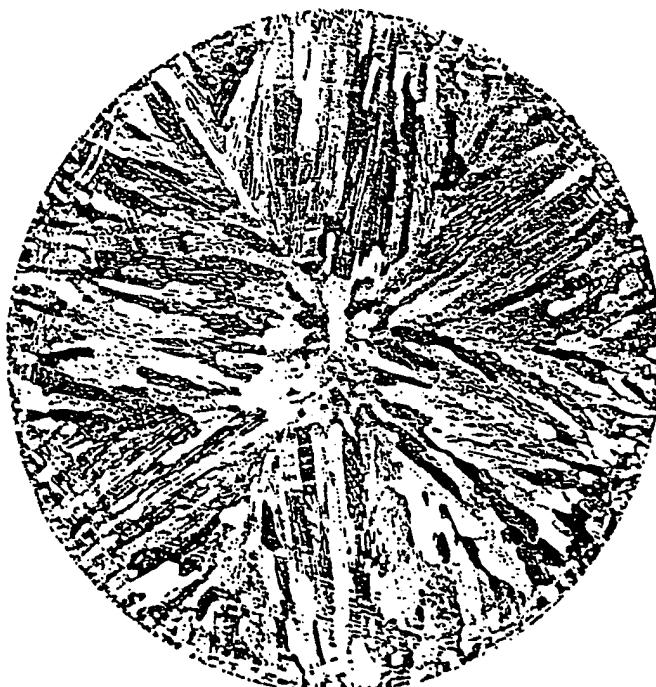
Fig. 6—Chill Casting of Nickel (X1).
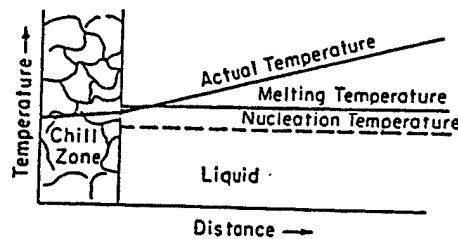
Fig. 7—Thermal Conditions During Freezing in
the Columnar Zone Of A Pure Metal.

in the chill zone heats the thermally supercooled liquid to the melting temperature. The thickness of the chill zone is determined by the extent of the supercooled region before nucleation occurs. This in turn depends upon the superheat in the liquid, the temperature of the mold, the thermal properties of the metal and the mold, and the nucleation potency of catalysts in the liquid or on the mold. In most instances nuclei form on the mold surface only, and the chill zone is limited to the surface grains in the casting as shown in Fig. 6, a section taken from a chill casting of nickel.

undeformed configuration at sufficiently large loads. Several types of bifurcation are found to occur. Explicit conditions determining the type of bifurcation are established for the general transversely isotropic composite sphere. In particular, if each phase is described by an explicit material model which may be viewed as a generalization of the classic neo-Hookean model to anisotropic materials, phenomena which were *not* observed for the homogeneous anisotropic sphere [1] nor for the composite neo-Hookean sphere (Horgan and Pence, 1989) may occur. The stress distribution as well as the possible prevention of interface debonding due to cavitation are also examined for the general composite sphere.

The effects of *material compressibility* were investigated in [3]. The absence of the zero volume change constraint complicates the analytical solution of problems, so that until quite recently, the major advances in nonlinear elasticity theory were made for incompressible materials. In earlier work a scheme was developed for simplifying the analytical solution of axisymmetric elastostatic problems for nonlinearly elastic compressible materials. It was shown how the *second-order* nonlinear ordinary differential equation arising from equilibrium could be reduced to a pair of *first-order* equations. This led to new analytical solutions (in closed form) for a variety of compressible material models. Another aspect of compressible materials undergoing large deformations was investigated in [3] following on earlier results published in 1992. The basic issue examined is to identify the class of compressible materials for which volume preserving (i.e. isochoric) anti-plane shear or azimuthal (or circular) shear deformations can occur in hollow circular tubes. (The anti-plane shear deformations are of concern in Mode III fracture of materials). It was shown in [3] for the azimuthal shear problem that a variety of compressible material models can sustain such volume preserving deformations. The results have implications for the design of experiments and for constitutive modeling.

In [4], we discuss the problem of obtaining upper and lower bounds for the strain-energy density in linear anisotropic elastic materials. One set of bounds is given in terms of the *magnitude of the stress field,* another in terms of the *magnitude of the strain field. Results of this kind play a major* role in the analysis of Saint-Venant's Principle for anisotropic materials and structures. They are also useful in estimating global quantities such as total energies, buckling loads, and natural frequencies. For several classes of elastic symmetry (e.g. cubic, transversely isotropic, hexagonal, and tetragonal symmetry) the *optimal constants* appearing in these bounds are given *explicitly* in terms of the elastic constants. This makes the results directly accessible to the design engineer. Such explicit results are rare in the field of anisotropic elasticity. For more elaborate symmetries (e.g., orthotropic, monoclinic, and triclinic) the optimal constants depend on the solution of cubic and sextic equations, respectively.

In [5] the linear theory of elasticity is used to study an homogeneous anisotropic semi-infinite strip, free of tractions on its long sides and subject to edge loads or displacements that produce stresses that decay in the axial direction. If one seeks solutions for the (dimensionless) Airy stress function of the form $\phi = e^{-\gamma x} F((y))$, γ constant, then one is led to a fourth-order-eigenvalue problem for $F(y)$ with complex eigenvalue γ. This problem, considered previously by Choi & Horgan (1977), is the anisotropic analog of the eigenvalue problem for the Fadle-Papkovich eigenfunctions arising in the isotropic case. The decay rate for Saint-Venant end effects is given by the eigenvalue with smallest positive real part. For an isotropic strip where the material is described by two elastic constants (Young's modulus and Poisson's ratio), the associated eigencondition is *independent* of these constants. For transversely isotropic (or specially orthotropic) materials, described by *four* elastic constants, the eigencondition depends only on *one* elastic parameter. Here, we treat the fully anisotropic strip described by *six* elastic constants and show that the eigencondition depends on only *two* elastic parameters. Tables and graphs for a scaled complex-valued eigenvalue are presented. These data allow one to determine the Saint-Venant decay length for the *fully anisotropic* strip, as we illustrate by a numerical example for an end-loaded off-axis graphite-epoxy strip.

The result obtained in [5] are presented in a form immediately accessible to design engineers. Thus, *for any arbitrary degree of anisotropy,* (or equivalently, for any off-axis orientation of a fiber-reinforced strip), the numerical data presented in [5] allow one to determine the Saint-Venant decay length precisely. The results should have widespread application to structural mechanics issues such as assessing end constraint problems in mechanical testing, determining the influence of fasteners, joints, cut-outs, etc. in composite structures and evaluation of the limits of strength-of-materials formulas when applied to composites.

In [6] we are concerned with assessing the effects of small perturbations in the constitutive laws on antiplane shear deformation fields arising in the theory of nonlinear elasticity. The mathematical problem is governed by a second-order quasilinear partial differential equation in divergence form. Displacement (or traction) boundary-value problems on a semi-infinite strip, with non-zero data on one end only, are considered. Such problems arise in investigation of Saint-Venant end effects in elasticity theory. The main result established provides a comparison between two solutions, one of which is a solution to a simpler equation, for example Laplace's equation. Three examples involving perturbations of power-law material models are used to illustrate the results. The results obtained are of interest in view of the practical difficulty in constructing constitutive models which provide an *exact description of material behavior.* Thus errors made in constructing a specific constitutive model are correlated with corresponding errors in the solution of boundary-value problems. Such results can be used by designers to

assess the robustness of particular constitutive models.

In [7], the effects of nonlinear boundary conditions on the decay of end effects in three-dimensional cylindrical bodies are investigated for steady state heat conduction. The nonlinear heat-loss or heat gain type boundary conditions are shown to significantly alter the decay behavior. For example, polynomial growth or decay, or exponential growth or decay can occur depending on the form of the nonlinearity. In [8] the effects of material inhomogeneity are examined for anti-plane shear deformations.

In [9], plane deformations of a rectangular strip, composed of an homogeneous fully anisotropic linearly elastic material, are considered. The strip is in equilibrium under the action of end loads, with the lateral sides traction-free. Two conservation properties for certain cross-sectional stress measures are established, generalizing previously known results for the isotropic case. It is noteworthy that in the first of these conservation laws only one of the off-axis elastic constants appears explicitly while in the second only the opposite off-axis constant appears explicitly. Such conservation properties are useful in assessing the influence of material anisotropy on Saint-Venant's principle, as well as in establishing convexity properties for cross-sectional stress measures. In particular, it is anticipated that the results should be useful in determining the extent of edge effects in the off-axis testing of anisotropic and composite materials.

In the context of linear elasticity theory, Saint-Venant's Principle is often used to justify the neglect of edge effects when determining stresses in a material. This is valid in the case of an isotropic material. However for the more general anisotropic material, experimental results have shown that edge effects may persist much farther into the material than in the isotropic case and cannot be neglected. In [11] the effect of material anisotropy on the exponential decay rate for stresses in a semi-infinite elastic strip is examined. A linear elastic semi-infinite strip in a state of plane stress/strain subject to a self-equilibrated end load is considered first for an orthotropic material and then for the most general anisotropic material. The problem is governed by a fourth-order elliptic partial differential equation with constant coefficients. Conservation properties of the solution are derived to help in determining decay rate estimates. Energy methods are then used to provide lower bounds on the actual decay rate. Both analytic and numerical estimates are obtained in terms of the elastic constants of the material and results are shown for a set of specific materials. When compared with the exact decay rate computed numerically from the eigenvalues of a fourth-order ordinary differential equation, the results in some cases show a high degree of accuracy not achieved previously. Results of the type obtained here have several practical applications, for example, in the mechanical testing of anisotropic and composite materials and in assessing the influence of fasteners, joints, etc. in composite structures.

Explicit analytic results of the type described above are crucial to the complete analysis of *local* or *end effects* in anisotropic or composite materials and structures. Previous work has shown that such end effects decay *much more slowly* than in isotropic materials. For transversely-isotropic (or specially orthotropic) materials, our earlier work has led to *specific design formulas* for the distance beyond which Saint-Venant edge effects can be neglected. The results, which have important implications for the experimental techniques used to measure material properties, have led to modifications of the ASTM standard test and are now quoted routinely in text- and hand-books on mechanics of composite materials. Our current research deals with more complicated degrees of anisotropy, including the general anisotropic case relevant to the off-axis tension test, and with effects of nonlinearity. Analytic results of the type obtained are *crucial complements to large-scale computational analyses*.

Anti-plane shear deformations of a cylindrical body, with a single displacement field parallel to the generators of the cylinder and independent of the axial coordinate, are one of the simplest classes of deformations that solids can undergo. They may be viewed as complementary to the more familiar plane deformations. Anti-plane (or longitudinal) shear deformations have been the subject of considerable recent interest in *nonlinear* elasticity theory for homogeneous isotropic solids. In contrast, for the *linear* theory of *isotropic* elasticity, such deformations are usually not extensively discussed. The purpose of the paper [12] is to demonstrate that for *inhomogeneous anisotropic linearly elastic solids* the anti-plane shear problem *does* provide a particularly tractable and illuminating setting within which effects of anisotropy and inhomogeneity may be examined. We consider infinitesimal anti-plane shear deformations of an inhomogeneous anisotropic linearly elastic cylinder subject to prescribed surface tractions on its lateral boundary whose only nonzero component is axial and which does not vary in the axial direction. In the absence of body forces, *not all arbitrary anisotropic cylinders* will sustain an anti-plane shear deformation under such tractions. Necessary and sufficient conditions on the elastic moduli are obtained which *do allow* an anti-plane shear. The resulting boundary value problems governing the axial displacement are formulated. The most general elastic symmetry consistent with an anti-plane shear is described. There are at most 15 independent elastic coefficients associated with such a material. In general, there is a *normal* axial stress present, which can be written as a linear combination of the two dominant shear stresses. For a material with the cylindrical cross-section a plane of elastic symmetry (monoclinic, with 13 moduli) the normal stress is no longer present. For *homogeneous* materials, it is shown how the governing boundary-value problem can be transformed to an equivalent isotropic problem for a transformed cross-sectional domain. Applications to the issue of assessing the influence of anisotropy and inhomogeneity on the decay of Saint-Venant end effects are described.

A comprehensive review of anti-plane shear for linear and nonlinear elasticity has been given in [13].

2. Personnel Supported: C. O. Horgan (P.I.)

Dr. Debra A. Polignone, former Ph.D. student, has been actively involved in this research. Her Ph.D. Dissertation [10], which resulted in publications [1, 2] was defended on May 3, 1993. Dr. Polignone was awarded a National Defense Science and Engineering Graduate Fellowship (NDSEGF) for 1990-1993, sponsored by the U.S. Air Force. Dr. Polignone was a post-doctoral Fellow at Center for Nonlinear Analysis, Carnegie-Mellon University, Pittsburgh, PA 1993-94. (Fellowship awarded as a result of national competition). She began a tenure-track Assistant Professorship at Dept. of Mathematics, University of Tennessee, Knoxville, June 1994. Dr. Polignone and Professor Horgan have now co-authored seven journal publications.

A second student, Dr. Kristin L. Miller, defended her Ph.D. Dissertation [11] on April 25, 1994. She was also awarded a NDSEGF, sponsored by U.S. Air Force, for 1991-1994. Her work resulted in publications [9, 12] and two forthcoming papers. She was awarded a NRC Postdoctoral fellowship in The Computational and Applied Mathematics Division, NIST for 1994-1996. She decided to accept a permanent position as Cryptologic Mathematician, National Security Agency (NSA), Fort Meade, MD, Sept. 1994. A third student, Sarah C. Baxter defended her Ph.D. Dissertation on April 5, 1995. Her work was supported in part by grants from the Virginia Space Grant Consortium and ARO. During 1995-96, she will be a Post-Doctoral Research Fellow in the Department of Civil Engineering, University of Virginia.

3. Awards

1) C. O. Horgan was elected a member of the Board of Directors, Society of Engineering Science, Inc., October 1993 in recognition of research and administrative achievements..

2) Invited keynote speaker at ARO Workshop entitled "Dynamic Response of Composite Structures," University of New Orleans, August 1993.

3) Research highlighted by Structures and Dynamics Program, ARO, as one of the most outstanding research projects funded by the Program in 1993.

4) C. O. Horgan was named the Wills Johnson Professor of Applied Mathematics & Mechanics, University of Virginia, July 1, 1994.

4. Publications

(List of recent papers, theses and dissertations that have been supported, or partially supported, by AFOSR.)

1. Polignone, Debra A., and Horgan, C. O., "Cavitation for Anisotropic Incompressible Non-linearly Elastic Spheres," *J. of Elasticity* **33**, 1993, 27-65.

Abstract. In this paper, the effect of *material anisotropy* on void nucleation and growth in *incompressible* nonlinearly elastic solids is examined. A bifurcation problem is considered for a solid sphere composed of an incompressible homogeneous nonlinearly elastic material which is transversely isotropic about the radial direction. Under a uniform radial tensile dead-load, a branch of radially symmetric configurations involving a traction-free internal cavity bifurcates from the undeformed configuration at sufficiently large loads. Closed form analytic solutions are obtained for a specific material model, which may be viewed as a generalization of the classic neo-Hookean model to anisotropic materials. In contrast to the situation for a neo-Hookean sphere, bifurcation here may occur locally either to the right (supercritical) or to the left (subcritical), depending on the degree of anisotropy. In the latter case, the cavity has finite radius on first appearance. Such a discontinuous change in stable equilibrium configurations is reminiscent of the snap-through buckling phenomenon of structural mechanics. Such dramatic cavitational instabilities were previously encountered by Antman and Negron-Marrero (1987) for anisotropic *compressible* solids and by Horgan and Pence (1989) for *composite* incompressible spheres.

2. Polignone, Debra A., and Horgan, C. O., "Effects of material anisotropy and inhomogeneity on cavitation for composite incompressible anisotropic nonlinearly elastic spheres," *Int. J. of Solids & Structures* **30**, 1993, 3381-3416.

Abstract. The effects of *material anisotropy* and *inhomogeneity* on void nucleation and growth in incompressible anisotropic nonlinearly elastic solids are examined. A bifurcation problem is considered for a composite sphere composed of two arbitrary homogeneous incompressible nonlinearly elastic materials which are transversely isotropic about the radial direction, and perfectly bonded across a spherical interface. Under a uniform radial tensile dead-load, a branch of radially symmetric configurations involving a traction-free internal cavity bifurcates from the undeformed configuration at sufficiently large loads. Several types of bifurcation are found to occur. Explicit conditions determining the type of bifurcation are established for the general transversely isotropic composite sphere. Dramatic cavitational instabilities reminiscent of the snap-through buckling phenomena of structural mechanics are found to occur. In particular, if each phase is described by an explicit material model which may be viewed as a

generalization of the classic neo-Hookean model to anisotropic materials, phenomena which were not observed for the homogeneous anisotropic sphere [Antman and Negron-Marrero, 1987, Polignone and Horgan, 1993] nor for the composite neo-Hookean sphere [Horgan and Pence, 1989] may occur. The stress distribution as well as the possible role of cavitation in preventing interface debonding are also examined for the general composite sphere.

3. Polignone, Debra A., and Horgan, C. O., "Pure Azimuthal Shear of Compressible Non-Linearly Elastic Circular Tubes," *Quarterly of Applied Mathematics,* **52**, 1994, 113-131.

Abstract. We consider azimuthal (or circular) *shear* of a hollow circular cylinder, composed of a homogeneous isotropic *compressible* nonlinearly elastic material. The inner surface of the tube is bonded to a rigid cylinder. The deformation may be achieved either by applying a uniformly distributed azimuthal shear traction on the outer surface together with zero radial traction (Problem 1) *or* by subjecting the outer surface to a prescribed angular displacement, with zero radial displacement (Problem 2). For an arbitrary compressible material, the cylinder will undergo both a radial and angular deformation. The class of materials for which *pure azimuthal shear* (i.e. a deformation with zero radial displacement) is possible is described. The corresponding angular displacement and stresses are determined explicitly. Specific material models are used to illustrate the results.

4. Mehrabadi, M. M., Cowin, S. C. and Horgan, C. O., "Strain Energy Density Bounds for Linear Anisotropic Elastic Materials," *Journal of Elasticity* **30** (1993), 191-196.

Abstract. Upper and lower bounds are presented for the magnitude of the strain energy density in linear anisotropic elastic materials. One set of bounds is given in terms of the magnitude of the stress field, another in terms of the magnitude of the strain field. Explicit algebraic formulas are given for the bounds in the case of cubic, transversely isotropic, hexagonal and tetragonal symmetry. In the case of orthotropic symmetry the explicit bounds depend upon the solution of a cubic equation, and in the case of the monoclinic and triclinic symmetries, on the solution of sixth order equations.

5. Crafter, E. C., Heise, R. M., Horgan, C. O., and Simmonds, J. G., "The eigenvalues for a self-equilibrated, semi-infinite, anisotropic elastic strip," *Journal of Applied Mechanics* (Trans. ASME), **60** (1993), 276-281.

Abstract The linear theory of elasticity is used to study an homogeneous anisotropic semi-infinite strip, free of tractions on its long sides and subject to edge loads or displacements that produce stresses that decay in the axial direction. If one seeks solutions for the (dimensionless) Airy stress function of the form $\phi = e^{-\gamma x} F(y)$, γ constant, then one is led to a fourth-order

eigenvalue problem for $F(y)$ with complex eigenvalues γ. This problem, considered previously by Choi & Horgan (1977), is the anisotropic analog of the eigenvalue problem for the Fadle-Papkovich eigenfunctions arising in the isotropic case. The decay rate for Saint-Venant end effects is given by the eigenvalue with smallest positive real part. For an isotropic strip, where the material is described by two elastic constants (Young's modulus and Poisson's ratio), the associated eigencondition is *independent* of these constants. For transversely isotropic (or specially orthotropic) materials, described by *four* elastic constants, the eigencondition depends only on *one* elastic parameter. Here, we treat the fully anisotropic strip described by *six* elastic constants and show that the eigencondition depends on only *two* elastic parameters. Tables and graphs for a scaled complex-valued eigenvalue are presented. These data allow one to determine the Saint-Venant decay length for the *fully anisotropic* strip, as we illustrate by a numerical example for an end-loaded off-axis graphite-epoxy strip.

6.   Horgan, C. O., and Payne, L. E., "The effect of constitutive law perturbations on finite anti-plane shear deformations of a semi-infinite strip," *Quarterly of Applied Mathematics* **51** (1993) 441-465.

Abstract. This paper is concerned with assessing the effects of small perturbations in the constitutive laws on anti-plane shear deformations fields arising in the theory of nonlinear elasticity. The mathematical problem is governed by a second-order quasilinear partial differential equation in divergence form. Dirichlet (or Neumann) boundary value problems on a semi-infinite strip, with non-zero data on one end only, are considered. Such problems arise in investigation of Saint-Venant end effects in elasticity theory. The main result established provides a comparison between two solutions, one of which is a solution to a simpler equation, for example Laplace's equation. Three examples involving perturbation of power-law material models are used to illustrate the results.

7.   Horgan, C. O. and Payne, L. E., "Phragmén-Lindelöf type results for harmonic functions with nonlinear boundary conditions," *Archive for Rational Mechanics and Analysis* **122** (1993), 123-144.

Abstract. We are concerned with investigating the asymptotic behavior of harmonic functions defined on a three-dimensional semi-infinite cylinder, where homogeneous nonlinear boundary conditions are imposed on the lateral surface of the cylinder. Such problems arise in the theory of steady-state heat conduction. The classical Phragmén-Lindelöf theorem states that harmonic functions which vanish on the lateral surface of the cylinder must either grow exponentially or decay exponentially with distance from the finite end of the cylinder. Here we show

that the results are significantly different when the homogeneous Dirichlet boundary condition is replaced by the nonlinear heat-loss or heat-gain type boundary condition. We show that *polynomial growth (or decay)* or *exponential growth (or decay)* may occur, depending on the form of the nonlinearity. Explicit estimates for the growth or decay are obtained.

8. Horgan, C. O. and Payne, L. E., "On the asymptotic behavior of solutions of linear second-order boundary-value problems on a semi-infinite strip," *Archive for Rational Mechanics and Analysis* **124** (1993), 277-303.

Abstract. This paper is concerned with investigating the asymptotic behavior of solutions of a linear second-order variable coefficient elliptic partial differential equation in divergence form defined on a two-dimensional semi-infinite strip. Such problems arise in the theory of steady-state heat conduction for inhomogeneous anisotropic materials as well as in the theory of anti-plane shear deformations for a linearized inhomogeneous anisotropic elastic solid. For solutions of such equations which vanish on the long sides of the strip, a theorem of Phragmén-Lindelöf type is established providing estimates for the rate of growth or decay which are optimal for the case of constant coefficients. The results are illustrated by several examples. The estimates obtained in this paper can be used to assess the influence of inhomogeneity and anisotropy on the decay of end effects arising in connection with Saint-Venant's principle.

9. Miller, K. L. and Horgan, C. O., "Conservation Properties for Plane Deformations of Isotropic and Anisotropic Linearly Elastic Strips," *Journal of Elasticity* **33** (1993), 311-318.

Abstract. Plane deformations of a rectangular strip, composed of an homogeneous fully anisotropic linearly elastic material, are considered. The strip is in equilibrium under the action of end loads, with the lateral sides traction-free. Two conservation properties for certain cross-sectional stress measures are established, generalizing previously known results for the isotropic case. It is noteworthy that in the first of these conservation laws only one of the off-axis elastic constants appears explicitly while in the second only the opposite off-axis constant appears explicitly. Such conservation properties are useful in assessing the influence of material anisotropy on Saint-Venant's principle, as well as in establishing convexity properties for cross-sectional stress measures. In particular, it is anticipated that the results should be useful in determining the extent of edge effects in the off-axis testing of anisotropic and composite materials.

10. Polignone, Debra A., "A Bifurcation Approach Modeling Cavitation in Anisotropic Non-linearly Elastic Solids," Ph.D. Dissertation, May 3, 1993.

Abstract. The fundamental mechanics and mathematics of *void nucleation and growth* due to large deformations in *anisotropic* nonlinearly elastic solids are explored using the model of

finite elasticity. Compared to the isotropic case, very little is known. Mathematically, investigation of singular solutions of the second-order quasilinear system of partial differential equations describing equilibrium states of nonlinearly elastic anisotropic bodies is involved. For radially symmetric deformations, the basic problem reduces to a bifurcation problem for a single second-order nonlinear ordinary differential equation. The possibility for discontinuous changes in stable equilibrium configurations (reminiscent of snap-through buckling in structural mechanics) is explored. Analysis for both homogeneous and composite spheres is carried out.

11. Miller, K. L., "End Effects for Plane Deformations of an Elastic Anisotropic Semi-Infinite Strip," Ph.D. Dissertation, April 25, 1994.

Abstract. In the context of linear elasticity theory, Saint-Venant's Principle is often used to justify the neglect of edge effects when determining stresses in a body. This is valid in the case of an isotropic material. However for the more general anisotropic material, experimental results have shown that edge effects may persist much farther into the material than in the isotropic case and cannot be neglected. This research examines the effect of material anisotropy on the exponential decay rate for stresses in a semi-infinite elastic strip. A linear elastic semi-infinite strip in a state of plane stress/strain subject to a self-equilibrated end load is considered first for an orthotropic material and then for the most general anisotropic material. The problem is governed by a fourth-order elliptic partial differential equation with constant coefficients. Conservation properties of the solution are derived to help in determining decay rate estimates. Energy methods are then used to provide lower bounds on the actual decay rate. Both analytic and numerical estimates are obtained in terms of the elastic constants of the material and results are shown for a set of specific materials. When compared with the exact decay rate computed numerically from the eigenvalues of a fourth-order ordinary differential equation, the results in some cases show a high degree of accuracy not achieved previously. Results of the type obtained here have several practical applications, for example, in the mechanical testing of anisotropic and composite materials and in assessing the influence of fasteners, joints, etc. in composite structures.

12. C. O. Horgan and K. L. Miller, "Anti-plane shear deformations for homogeneous anisotropic linearly elastic solids," *J. Applied Mechanics* **61** (1994) 23-29.

Abstract. Anti-plane shear deformations of a cylinder body, with a single displacement field parallel to the generators of the cylinder and independent of the axial coordinate, are one of the simplest classes of deformations that solids can undergo. They may be viewed as complementary to the more familiar plane deformations. Anti-plane (or longitudinal) shear deformations have been the subject of considerable recent interest in *nonlinear* elasticity theory for

homogenous isotropic solids. In contrast, for the *linear* theory of *isotropic* elasticity, such deformations are usually not extensively discussed. The purpose of the paper [8] is to demonstrate that for *inhomogenous anisotropic linearly elastic solids* the anti-plane shear problem *does* provide a particularly tractable and illuminating setting within which effects of anisotropy and inhomogeneity may be examined. We consider infinitesimal anti-plane shear deformations of an inhomogenous anisotropic linearly elastic cylinder subject to prescribed surface tractions on its lateral boundary whose only nonzero component is axial and which does not vary in the axial direction. In the absence of body forces, *not all arbitrary anisotropic cylinders will sustain an* anti-plane shear deformation under such tractions. Necessary and sufficient conditions on the elastic moduli are obtained which *do allow* an anti-plane shear. The resulting boundary value problems governing the axial displacement are formulated. The most general elastic symmetry consistent with an anti-plane shear is described. There are at most 15 independent elastic coefficients associated with such a material. In general, there is a *normal* axial stress present, which can be written as a linear combination of the two dominant shear stresses. For a material with the cylindrical cross-section a plane of elastic symmetry (monoclinic, with 13 moduli) the normal stress is no longer present. For *homogenous* materials, it is shown how the governing boundary-value problem can be transformed to an equivalent isotropic problem for a transformed cross-sectional domain. Applications to the issue of assessing the influence of anisotropy and inhomogeneity on the decay of Saint-Venant end effects are described.

13. C. O. Horgan, "Anti-plane shear deformations in linear and nonlinear solid mechanics," *SIAM Review* **37** (1995), 53-81.

Abstract. The intent of this expository paper is to draw the attention of the applied mathematics community to an interesting two-dimensional mathematical model arising in solid mechanics involving a single second-order linear or quasilinear partial differential equation. This model has the virtue of relative mathematical simplicity without loss of essential physical relevance. *Anti-plane* shear deformations are one of the simplest classes of deformations that solids can undergo. In anti-plane shear (or longitudinal shear, generalized shear) of a cylindrical body, the displacement is parallel to the generators of the cylinder and is independent of the axial coordinate. Thus anti-plane shear, with just a single scalar axial displacement field, may be viewed as complementary to the more complicated (yet perhaps more familiar) plane strain deformation, with its two in-plane displacements. In recent years, considerable attention has been paid to the analysis of anti-plane shear deformations within the context of various constitutive theories (linear and nonlinear) of solid mechanics. Such studies were largely motivated by the promise of relative analytic simplicity compared with plane problems since the governing

equations are a single second-order linear or quasilinear partial differential equation rather than higher-order or coupled systems of partial differential equations. Thus, the anti-plane shear problem plays a useful role as a pilot problem, within which various aspects of solutions in solid mechanics may be examined in a particularly simple setting.

In this review article, modern developments concerning the anti-plane shear model and its applications are described for both linear and nonlinear solid mechanics.

5. Presentations, DoD contact, Technology Transfer

1. "Cavitation for incompressible anisotropic nonlinearly elastic spheres," (C. O. Horgan and Debra A. Polignone). Invited lecture at Special Session on "Recent Developments in Nonlinear Elasticity II," 30th Annual Meeting, Society of Engineering Science (joint with ASME + ASCE), Univ. of Virginia, Charlottesville, VA, June 1993. (Invited.)

2. "The eigenvalues for a self-equilibrated, semi-infinite, anisotropic elastic strip" (E. C. Crafter, R. M. Heise, C. O. Horgan, and J. G. Simmonds). Presented at Session "Elasticity II," 30th Annual Meeting, Society of Engineering Science (joint with ASME & ASCE), Univ. of Virginia, Charlottesville, VA, June 1993.

3. "End effects in anisotropic and composite structures" (C. O. Horgan and J. G. Simmonds). Presented at Army Symposium on Solid Mechanics 1993, Plymouth, Mass., August 1993.

4. "Saint-Venant end effects in composite structures" (C. O. Horgan). Invited keynote address (45 mins.) at ARO Workshop entitled "Dynamic Response of Composite Structures," University of New Orleans, August 1993.

5. "Cavitation for incompressible anisotropic nonlinearly elastic spheres," (with Debra A. Polignone). Invited lecture at Special Session on "Nonlinear Elasticity," 23rd Midwestern Mechanics Conference, Univ. of Nebraska, Lincoln, October 1993. (Invited.)

6. "A bifurcation approach to modeling void nucleation and growth in solids," Invited colloquium lecture at the Laboratory for Applied and Computational Mathematics, National Institute for Standards and Technology (NIST), Gaithersburg, MD, October 1993.

7. "Bifurcation and cavitation in anisotropic nonlinear elasticity" (with Debra A. Polignone). Invited lecture at Symposium on "Bifurcation Problems in Nonlinear Elasticity," ASME Winter Annual Meeting, New Orleans, November 1993. (Invited.)

8. "Effects of material anisotropy and inhomogeneity on cavitation for composite incompressible anisotropic nonlinearly elastic spheres" (with Debra A. Polignone). SIAM Conference on "Emerging Issues in Mathematics and Computation from the Materials Sciences," Pittsburgh, PA, April 1994.

9. "Effects of material anisotropy and inhomogeneity on cavitation for composite incompressible anisotropic nonlinearly elastic spheres" (with Debra A. Polignone). Invited lecture at Special Session on "Constitutive Formulations and Applications in Bioelasticity and Nonlinear Elasticity," 12th U.S. National Congress of Applied Mechanics, University of Washington, Seattle, June 1994.

10. "Anti-Plane Shear in Linear Anisotropic Elasticity" (C. O. Horgan). Invited lecture at Special Session on "Multiphase Elasticity and the Dundurs parameters," 12th U.S. National Congress of Applied Mechanics, University of Washington, Seattle, June 1994.

11. "A bifurcation approach to modeling void nucleation and growth in solids" (C. O. Horgan). Invited lecture at the Flight Dynamics Laboratory, Wright Patterson Air Force Base, Ohio, August 1994.

12. "On axisymmetric solutions in compressible nonlinearly elastic solids" (C. O. Horgan). Invited lecture at Special Session on "Elasticity," 31st Annual Meeting, Society of Engineering Science, Texas A & M University, October 1994.

13. "Axisymmetric solutions for compressible nonlinear elastic solids." Invited lecture at Special Session on "Elasticity," 15th Canadian Congress of Applied Mechanics, CANCAM95, University of Victoria, B.C., CANADA, May 1995.

14. Research visit to Boeing Aircraft Company, Seattle, WA, July 1st 1994, May 25th 1995 (C. O. Horgan).

15. "Cavitational failure mechanisms in rubber." Invited lecture at AFOSR Workshop on "Structural Analysis of Shells with Application to Aircraft Tire Performance and Service Life," Flight Dynamics Directorate, Wright Patterson AFB, Ohio, July 27, 1995.

16. Several colloquium lectures delivered at various universities.

17. Results of research widely distributed to NSF, ARO, AFOSR, NIST, and private industry.

18. Continuous contact maintained with Dr. A. Nachman, Program Director, Applied Analysis, AFOSR via telephone, preprint and reprint submission, AFOSR Annual Data.